

MIXED FROBENIUS STRUCTURE AND LOCAL QUANTUM COHOMOLOGY

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ABSTRACT. In a previous paper, the authors introduced the notion of mixed Frobenius structure (MFS) as a generalization of the structure of a Frobenius manifold. Roughly speaking, the MFS is defined by replacing a metric of the Frobenius manifold with a filtration on the tangent bundle equipped with metrics on its graded quotients. The purpose of the current paper is to construct a MFS on the cohomology of a smooth projective variety whose multiplication is the non-equivariant limit of the quantum product twisted by a concave vector bundle. We show that such a MFS is naturally obtained as the non-equivariant limit of the Frobenius structure in the equivariant setting.

1. INTRODUCTION

We continue our study on mixed Frobenius structure and local quantum cohomology initiated in [8].

1.1. A mixed Frobenius algebra. Let K be a field. A finite-dimensional associative commutative K -algebra A equipped with a non-degenerate bilinear form g (called a *metric*) is called a Frobenius algebra if g is invariant under the product, i.e., $g(xy, z) = g(x, yz)$ holds for any $x, y, z \in A$.

In [8], the following generalization of the Frobenius algebra was introduced. Let A be a K -algebra as above. By definition, a Frobenius filtration (I_\bullet, g_\bullet) on A consists of an exhaustive increasing filtration I_\bullet by ideals and A -invariant metrics g_\bullet on its graded quotients (Definition 2.1). We call an algebra with a Frobenius filtration a mixed Frobenius algebra. If the filtration is trivial, this is nothing but the notion of Frobenius algebra. We show that any algebra over an algebraically closed field admits a Frobenius filtration (Theorem 2.3). This is in contrast to the fact that not all algebras admit invariant metrics.

One of main results of this paper is to show that a mixed Frobenius K -algebra appears in the limit as $\lambda \rightarrow 0$ of a “Frobenius algebra over $K[\lambda]$ ” (§3). The precise statement is as follows. Let H_K^λ be a free $K[\lambda]$ -module of finite rank equipped with a symmetric $K[\lambda]$ -bilinear form $g^\lambda : H_K^\lambda \times H_K^\lambda \rightarrow K[\lambda, \lambda^{-1}]$. If g^λ is unimodular over $K[\lambda, \lambda^{-1}]$, then it defines on the K -vector space $H_K := H_K^\lambda / \lambda H_K^\lambda$ an exhaustive increasing filtration by subspaces and metrics on its graded quotients (Lemma 3.4). We call such a pair a

nondegenerate filtration. Moreover if H_K^λ is equipped with a $K[\lambda]$ -algebra structure with respect to which g^λ is invariant, then the nondegenerate filtration is a Frobenius filtration on H_K with respect to the induced multiplication (Theorem 3.5). This construction is a generalization of the nilpotent construction [8, §3.1] (cf. §3.2).

1.2. A mixed Frobenius structure. A Saito structure (without a metric)¹ on a complex manifold M [11, §VII.1] is a triple consisting of a torsion-free flat affine connection ∇ , a symmetric Higgs field $\Phi : T_M \rightarrow \text{End } T_M$ and a vector field E called an Euler vector field satisfying certain compatibility conditions (see Definition 4.1). A symmetric Higgs field gives rise to a fiberwise commutative associative multiplication \circ on T_M . If a Saito structure (∇, Φ, E) on M is further equipped with a \circ -invariant metric g on T_M compatible with the other data, then the Saito structure (∇, Φ, E) with the metric g is equivalent to a Frobenius manifold structure on M [2].

Now we introduce the notion of the mixed Frobenius structure which generalizes the Frobenius manifold structure. The idea is to replace a \circ -invariant metric g with a Frobenius filtration (I_\bullet, g_\bullet) . Namely, we define a mixed Frobenius structure (MFS) on a manifold M to be a Saito structure (∇, Φ, E) on M together with a nondegenerate filtration (I_\bullet, g_\bullet) on the tangent bundle T_M subjecting to various compatibility conditions (Definition 4.5). In particular, it is required that $(T_M, \circ, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra. We arrived at this notion through our study on local mirror symmetry [1]. For detail about the motivation, we refer to [8, §1.3]. Notice that we slightly modify the definition of the MFS from [8] (cf. Remark 4.7).

For the application to the local quantum cohomology, it is necessary to consider a formal and logarithmic version of MFS. Let K be a subfield of \mathbb{C} . Let $R = K[[t_1, \dots, t_n, q_1, \dots, q_m]]$ and $M = \text{Spf } R$ be the formal completion of $K^{n+m} = \text{Spec } K[t, q]$ at the origin. We consider the logarithmic structure on M defined by the divisor $\{q_1 \cdots q_m = 0\}$ on K^{n+m} . We denote by M^\dagger the resulting logarithmic formal scheme. We define a formal (and logarithmic) MFS on M^\dagger in §5. As in the case of mixed Frobenius algebra, we show that a formal MFS on M^\dagger is obtained in the limit as $\lambda \rightarrow 0$ of a “formal Frobenius structure over $K[\lambda]$ ” on M^\dagger (Proposition 5.5).

1.3. MFS from local quantum cohomology. Let X be a smooth complex projective variety and let $H_{\mathbb{C}} := H^{\text{even}}(X, \mathbb{C})$. We choose a nef basis $\{\phi_1, \dots, \phi_p\}$ of $H^2(X, \mathbb{Z})$ and extend it to a homogenous basis $\{\phi_0 = 1, \phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_s\}$ of $H_{\mathbb{C}}$. Let t_0, \dots, t_s be the coordinates on $H_{\mathbb{C}}$ associated to the basis. We set $R = K[[t, q]]$ where $t = (t_0, t_{p+1}, \dots, t_s)$ and $q = (q_1, \dots, q_p)$ with $q_i = e^{t_i}$. Let M^\dagger be the logarithmic formal scheme defined as in the previous paragraph.

¹ In this article, we call a Saito structure without a metric a *Saito structure* for short.

Fix a concave holomorphic vector bundle \mathcal{V} on X (e.g., \mathcal{V} is the dual of an ample line bundle). We construct a formal MFS on M^\dagger from \mathcal{V} as follows. Let us introduce the fiberwise S^1 -action on \mathcal{V} by scalar multiplication. Then, following Givental [3], we consider the S^1 -equivariant Gromov–Witten invariants of X and the intersection pairing on X , both twisted by the inverse of the S^1 -equivariant Euler class of \mathcal{V} . Using them, one can define the twisted quantum cup product $*_{\mathcal{V}}$ on $H_{\mathbb{C}}^\lambda := H_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ where $\mathbb{C}[\lambda] = H_{S^1}^*(pt, \mathbb{C})$ is identified with the S^1 -equivariant cohomology of a point. Identifying the logarithmic tangent sheaf $\mathcal{T}_{M^\dagger}^\lambda := \mathcal{T}_{M^\dagger} \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ of M^\dagger over $\mathbb{C}[\lambda]$ with $\mathcal{O}_M \otimes_{\mathbb{C}} H_{\mathbb{C}}^\lambda$, we obtain a formal Frobenius structure over $\mathbb{C}[\lambda]$ on M^\dagger . Then, as an application of the results in §5, we obtain a formal MFS on M^\dagger in the non-equivariant limit (i.e. the limit as $\lambda \rightarrow 0$) (Theorem 6.4).

As we mentioned earlier, our motivation to study MFS comes from local mirror symmetry [1]. Relationships to *loc.cit.* and to our previous work [7] are explained in §6.4.

1.4. Conventions. (1) Let K be a field. A K -algebra means a finite-dimensional commutative associative K -algebra with a unit.

(2) Given a commutative ring R , an R -algebra structure on a free R -module means an associative commutative R -bilinear multiplication which admits a unit.

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2. MIXED FROBENIUS ALGEBRA

2.1. Frobenius filtration and mixed Frobenius algebra. Let K be a field. A nondegenerate symmetric bilinear form g on a K -vector space is called a *metric*. A pair (I_\bullet, g_\bullet) consisting of an exhaustive increasing filtration I_\bullet on a K -vector space by subspaces and a collection of metrics g_\bullet on $I_\bullet/I_{\bullet-1}$ is called a *nondegenerate filtration* on the vector space.

Let A be a K -algebra. We say that a metric g on an A -module I is A -invariant if it satisfies the condition

$$g(a \cdot x, y) = g(x, a \cdot y) \quad (a \in A, x, y \in I) .$$

Definition 2.1. A Frobenius filtration on a K -algebra A is a nondegenerate filtration (I_\bullet, g_\bullet) on A such that each filter I_\bullet is an ideal of A and the metric g_\bullet on $I_\bullet/I_{\bullet-1}$ is $A/I_{\bullet-1}$ -invariant.

Definition 2.2. A mixed Frobenius K -algebra is a pair which consists of a K -algebra A and a Frobenius filtration (I_\bullet, g_\bullet) on A .

2.2. Existence of Frobenius filtrations. In this subsection, the field K is assumed to be algebraically closed.

Theorem 2.3. *Any finite-dimensional K -algebra A has a Frobenius filtration.*

Let $\mathfrak{N} = \sqrt{0}$ be the nilradical of A . Note that the finite-dimensionality of A implies that \mathfrak{N} is a nilpotent ideal and that \mathfrak{N} coincides with the Jacobson radical of A . It follows from the latter that A/\mathfrak{N} is a semi-simple algebra. Consider the decreasing sequence of ideals $A \supset \mathfrak{N} \supset \mathfrak{N}^2 \supset \cdots \supset \mathfrak{N}^{l-1} \supset \mathfrak{N}^l = 0$.

Lemma 2.4. *$\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is a completely reducible A/\mathfrak{N}^{i+1} -module.*

Proof. Consider the exact sequence of A -modules

$$0 \longrightarrow \mathfrak{N}/\mathfrak{N}^{i+1} \longrightarrow A/\mathfrak{N}^{i+1} \longrightarrow A/\mathfrak{N} \longrightarrow 0 .$$

It follows that A/\mathfrak{N}^{i+1} acts on $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ via A/\mathfrak{N} , since $\mathfrak{N}/\mathfrak{N}^{i+1}$ annihilates $\mathfrak{N}^i/\mathfrak{N}^{i+1}$. Then the semi-simplicity of A/\mathfrak{N} implies that $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is a completely reducible A/\mathfrak{N} -module, hence it is also a completely reducible A/\mathfrak{N}^{i+1} -module. \square

Lemma 2.5. *Let B be a finite-dimensional K -algebra. Then for any simple B -module $S \neq 0$, we have $\dim_K S = 1$.*

Proof. Since S is a simple B -module, there is a maximal ideal \mathfrak{m} of B such that $S \cong B/\mathfrak{m}$ as a B -module. The finite-dimensionality of B implies that the composition $K \rightarrow B \rightarrow B/\mathfrak{m}$ is a field extension of finite degree. It then follows that $B/\mathfrak{m} \cong K$, since K is algebraically closed. \square

Proof of Theorem 2.3. By the above two Lemmas, $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ is the direct sum of 1-dimensional simple A/\mathfrak{N}^{i+1} -modules. If we take a basis $x_{i,j}$ ($1 \leq j \leq \dim_K \mathfrak{N}^i/\mathfrak{N}^{i+1}$) of the simple modules and define a bilinear form $\langle \cdot, \cdot \rangle_i$ on $\mathfrak{N}^i/\mathfrak{N}^{i+1}$ by

$$\langle x_{i,j}, x_{i,k} \rangle_i = \delta_{j,k} ,$$

then $\langle \cdot, \cdot \rangle_i$ is an invariant metric. Thus the filtration $I_\bullet := \mathfrak{N}^{l-\bullet}$ with metrics $g_\bullet := \langle \cdot, \cdot \rangle_{i-\bullet}$ is a Frobenius filtration on A . \square

3. MIXED FROBENIUS ALGEBRA FROM A LOCALIZED $K[\lambda]$ -METRIC

3.1. Construction of mixed Frobenius algebra. Let K be a field and let H_K be a K -vector space of dimension s . We set $H_K^\lambda := H_K \otimes_K K[\lambda]$ and identify H_K with the quotient module $H_K^\lambda / \lambda H_K^\lambda$. Let $\pi : H_K^\lambda \rightarrow H_K = H_K^\lambda / \lambda H_K^\lambda$ be the projection.

Definition 3.1. A *localized $K[\lambda]$ -metric* on H_K^λ is a symmetric $K[\lambda]$ -bilinear form $g^\lambda : H_K^\lambda \times H_K^\lambda \rightarrow K[\lambda, \lambda^{-1}]$ which is unimodular over $K[\lambda, \lambda^{-1}]^2$.

Now assume that a localized $K[\lambda]$ -metric g^λ on H_K^λ is given. We will construct a nondegenerate filtration on H_K from g^λ .

Lemma 3.2. *There exist a pair of $K[\lambda]$ -module bases $\mathbf{x}_1, \dots, \mathbf{x}_s$ and $\mathbf{y}_1, \dots, \mathbf{y}_s$ of H_K^λ and a set of integers $\kappa_1 \geq \dots \geq \kappa_s$ satisfying*

$$(3.1) \quad g^\lambda(\mathbf{x}_i, \mathbf{y}_j) = \lambda^{-\kappa_i} \delta_{i,j}.$$

The integers κ_i 's are uniquely determined by g^λ (but not the bases).

Proof. Let G be the matrix representation of g^λ with respect to a $K[\lambda]$ -module basis of H_K^λ . Multiplying by λ^{k_0} with some $k_0 \in \mathbb{Z}$ if necessary, we assume that all entries of the matrix $\lambda^{k_0}G$ are polynomials. By the theorem of elementary divisors, $\lambda^{k_0}G$ can be transformed into a diagonal matrix by successive elementary transformations from the left and the right. This means that there exist $K[\lambda]$ -module bases $\{\mathbf{x}_i\}, \{\mathbf{y}_i\}$ of H_K^λ such that $\lambda^{k_0}g^\lambda(\mathbf{x}_i, \mathbf{y}_j) = \delta_{i,j} e_i$ where $e_1, \dots, e_s \in K[\lambda]$ are diagonal entries (i.e. the elementary divisors of $\lambda^{k_0}G$). The assumption of unimodularity over $K[\lambda, \lambda^{-1}]$ implies that e_i 's are monomials. \square

Let us define the sequence of $K[\lambda]$ -submodules:

$$(3.2) \quad I_k^\lambda = \{\mathbf{x} \in H_K^\lambda \mid \lambda^k g^\lambda(\mathbf{x}, \mathbf{y}) \in K[\lambda] \ (\forall \mathbf{y} \in H_K^\lambda)\} \quad (k \in \mathbb{Z}).$$

Concretely, I_k^λ is written as follows with the basis $\mathbf{x}_1, \dots, \mathbf{x}_s$ of Lemma 3.2.

$$(3.3) \quad I_k^\lambda = \bigoplus_{i:\kappa_i \leq k} K[\lambda] \mathbf{x}_i \oplus \bigoplus_{i:\kappa_i > k} \lambda^{\kappa_i - k} K[\lambda] \mathbf{x}_i.$$

The same formula holds for the other basis $\{\mathbf{y}_i\}$.

Lemma 3.3. *For $\mathbf{x}, \mathbf{y} \in I_k^\lambda$, $\text{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\mathbf{x}, \mathbf{y})$ depends only on $\pi(\mathbf{x}), \pi(\mathbf{y}) \in H_K$. Moreover $\text{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\mathbf{x}, \mathbf{y}) = 0$ if $\mathbf{x} \in I_{k-1}^\lambda$ or $\mathbf{y} \in I_{k-1}^\lambda$.*

Proof. Let us write $\mathbf{x}, \mathbf{y} \in I_k^\lambda$ as

$$\begin{aligned} \mathbf{x} &= \sum_{i:\kappa_i \leq k} f_i(\lambda) \mathbf{x}_i + \sum_{i:\kappa_i > k} \lambda^{\kappa_i - k} f_i(\lambda) \mathbf{x}_i \quad (f_i(\lambda) \in K[\lambda]), \\ \mathbf{y} &= \sum_{i:\kappa_i \leq k} h_i(\lambda) \mathbf{y}_i + \sum_{i:\kappa_i > k} \lambda^{\kappa_i - k} h_i(\lambda) \mathbf{y}_i \quad (h_i(\lambda) \in K[\lambda]). \end{aligned}$$

² This means that, given a $K[\lambda]$ -basis of H_K^λ , the representation matrix of g^λ is a unimodular matrix over $K[\lambda, \lambda^{-1}]$.

By eq.(3.1), we obtain

$$(3.4) \quad \text{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\mathbf{x}, \mathbf{y}) = \sum_{i: \kappa_i=k} f_i(0) h_i(0) .$$

The statement follows easily from this. \square

Let

$$(3.5) \quad I_k := \pi(I_k^\lambda) = \bigoplus_{i: \kappa_i \leq k} K \pi(\mathbf{x}_i) \quad (k \in \mathbb{Z}) .$$

By Lemma 3.3, the following bilinear form g_k on I_k/I_{k-1} is well-defined:

$$(3.6) \quad g_k(\bar{x}, \bar{y}) = \text{Res}_{\lambda=0} \lambda^{k-1} g^\lambda(\mathbf{x}, \mathbf{y}) \quad (x, y \in I_k) ,$$

where $x \mapsto \bar{x}$ denotes the projection $H_K \rightarrow H_K/I_{k-1}$ and \mathbf{x}, \mathbf{y} are any lifts of x, y to I_k^λ .

Lemma 3.4. (I_\bullet, g_\bullet) is a nondegenerate filtration on H_K .

Proof. Nondegeneracy of g_k follows from eq.(3.4). \square

Now assume that H_K^λ is equipped with an associative commutative $K[\lambda]$ -algebra structure $*$ with unit. Let g^λ be a localized $K[\lambda]$ -metric which is $*$ -invariant, i.e., g^λ satisfies

$$(3.7) \quad g^\lambda(\mathbf{x} * \mathbf{y}, \mathbf{z}) = g^\lambda(\mathbf{x}, \mathbf{y} * \mathbf{z}) \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in H^\lambda) .$$

On H_K , we have the induced multiplication and the nondegenerate filtration (I_\bullet, g_\bullet) defined in eqs.(3.5), (3.6).

Theorem 3.5. $(H_K, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra.

Proof. The $*$ -invariance (3.7) of g^λ implies that I_k^λ is an ideal. Therefore I_k is an ideal with respect to the induced multiplication \circ on H_K . The \circ -invariance of g_k follows from the $*$ -invariance of g^λ . \square

3.2. Nilpotent construction. Let (A, g) be a Frobenius K -algebra having nilpotent elements n_1, \dots, n_r and let

$$\mathbf{n} = \lambda^r + n_1 \lambda^{r-1} + \dots + n_r \in A[\lambda] .$$

As an example of Theorem 3.5, we consider the case $H_K^\lambda = A[\lambda]$ with the localized $K[\lambda]$ -metric g^λ given by

$$g^\lambda(\mathbf{x}, \mathbf{y}) := g(\mathbf{x} \cdot \mathbf{y}, \mathbf{n}^{-1}) = \sum_{j \geq 0} \frac{1}{\lambda^{(j+1)r}} g(\mathbf{x} \cdot \mathbf{y}, (\lambda^r - \mathbf{n})^j) \quad (\mathbf{x}, \mathbf{y} \in A[\lambda]) .$$

Let us calculate the nondegenerate filtration (I_\bullet, g_\bullet) defined by eqs.(3.5), (3.6). The ideals I_k^λ of $A[\lambda]$ defined in eq.(3.2) are given as follows.

Lemma 3.6. *We have*

$$(3.8) \quad I_k^\lambda = \begin{cases} \{\lambda^{-k} \mathbf{n} \cdot \mathbf{x} \mid \mathbf{x} \in A[\lambda]\} & (k \leq 0) \\ I_0^\lambda \oplus J_k^\lambda & (k > 0) \end{cases}.$$

In the last line, the direct sum is that of A -modules and

$$J_k^\lambda = \{\mathbf{x} \in A^{<r}[\lambda] \mid \lambda^k \mathbf{x} \text{ is divisible by } \mathbf{n} \},$$

where $A^{<r}[\lambda] = \{\mathbf{x} \in A[\lambda] \mid \deg \mathbf{x} < r\}$. Here $\deg \mathbf{x}$ is the degree of \mathbf{x} with respect to λ .

Proof. Since \mathbf{n} is monic of degree r , any $\mathbf{x} \in A[\lambda]$ is written uniquely as $\mathbf{x} = \mathbf{n} \cdot \mathbf{x}' + \mathbf{x}''$ with $\deg \mathbf{x}'' < r$.

First, we consider the case $k = 0$. It is easy to see that $\mathbf{x} \in I_0^\lambda$ if and only if $g^\lambda(\mathbf{x}'', y) = 0$ for any $y \in A$. If $\mathbf{x}'' = \sum_{i=1}^r a_i \lambda^{r-i}$ then we have,

$$g^\lambda(\mathbf{x}'', y) = \sum_{1 \leq i \leq r, j \geq 0} g(a_i (\lambda^r - \mathbf{n})^j, y) \lambda^{-(i+jr)} = \frac{g(a_1, y)}{\lambda} + o\left(\frac{1}{\lambda^2}\right).$$

From this equation, for $x \in I_0^\lambda$, it is necessary to have $g(a_1, y) = 0$ for any $y \in A$, hence $a_1 = 0$. Then we have

$$g^\lambda(\mathbf{x}'', y) = \frac{g(a_2, y)}{\lambda^2} + o\left(\frac{1}{\lambda^3}\right),$$

hence $a_2 = 0$. Repeating this process, we obtain $\mathbf{x}'' = 0$.

Next, we consider the case $k < 0$. If $\mathbf{x} \in I_k^\lambda$ then $\mathbf{x} \in I_0^\lambda$. Therefore it is written as $\mathbf{x} = \mathbf{n} \cdot \mathbf{x}'$. Since $\mathbf{x} \in I_k^\lambda$, we have $\lambda^k g^\lambda(\mathbf{x}, y) = \lambda^k g(\mathbf{x}', y) \in K[\lambda]$ for any $y \in A$. It then follows that the coefficients of \mathbf{x}' up to degree $-k - 1$ must be zero. Hence \mathbf{x}' is divisible by λ^{-k} .

For the case $k > 0$, it is easy to see that $\mathbf{x} \in I_k^\lambda$ if and only if $\lambda^k \mathbf{x}''$ is divisible by \mathbf{n} . \square

Let $N : A^{\oplus r} \rightarrow A^{\oplus r}$ be the homomorphism given by

$$N \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & 0 \\ -n_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n_{r-1} & 0 & 0 & \cdots & 1 \\ -n_r & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

The projections $A^{\oplus r} \rightarrow A$ to the first and the r th factors are denoted p_1, p_r .

Lemma 3.7. *We have*

$$(3.9) \quad I_k = \begin{cases} 0 & (k < 0) \\ \{x \cdot n_r \mid x \in A\} & (k = 0) \\ I_0 + J_k & (k > 0) \end{cases},$$

where $J_k = p_r(\text{Ker } N^k)$.

Proof. By Lemma 3.6, it is enough to show that $\pi(J_k^\lambda) = J_k$ for $k > 0$. Let $\rho : A^{<r}[\lambda] \rightarrow A^{\oplus r}$ be the isomorphism $\sum_{i=1}^r a_i \lambda^{r-i} \mapsto {}^t(a_1, \dots, a_r)$. Notice that $\rho^{-1} \circ N \circ \rho : A^{<r}[\lambda] \rightarrow A^{<r}[\lambda]$ maps \mathbf{x} to the remainder of $\lambda \mathbf{x}$ divided by \mathbf{n} . By induction on k , we can show that

$$(3.10) \quad \lambda^k \mathbf{x} = \sum_{i=0}^{k-1} (p_1 \circ N^i \circ \rho)(\mathbf{x}) \lambda^{k-1-i} \cdot \mathbf{n} + (\rho^{-1} \circ N^k \circ \rho)(\mathbf{x}) \quad (\mathbf{x} \in A^{<r}[\lambda]) .$$

Thus we obtain

$$J_k^\lambda = \{\mathbf{x} \in A^{<r}[\lambda] \mid \rho(\mathbf{x}) \in \text{Ker } N^k\} .$$

From this $\pi(J_k^\lambda) = J_k$ follows. □

Lemma 3.8. *We have*

$$\begin{aligned} g_0(x \cdot n_r, y \cdot n_r) &= g(x \cdot y, n_r) , \\ g_k(\bar{x}, \bar{y}) &= g(x, p_1(N^{k-1} \vec{y})) \quad (k > 0, x, y \in J_k) , \end{aligned}$$

where $\vec{y} \in \text{Ker } N^k$ is any lift of y satisfying $p_r(\vec{y}) = y$.

Proof. The case $k = 0$ is clear. To show the case $k > 0$, let $\mathbf{x}, \mathbf{y} \in J_k^\lambda$ be lifts of x, y . By eq.(3.10), we have

$$g_k(\bar{x}, \bar{y}) = g^\lambda\left(\mathbf{x}, \frac{\lambda^k \mathbf{y}}{\mathbf{n}}\right) \Big|_{\lambda=0} = g(x, p_1(N^{k-1} \rho(\mathbf{y}))) .$$

□

As a corollary of Theorem 3.5, we obtain

Proposition 3.9. *$(A, I_\bullet, g_\bullet)$ with I_\bullet, g_\bullet given in Lemmas 3.7, 3.8, is a mixed Frobenius algebra.*

When $r = 1$, $J_k = \{x \in A \mid n_1^k \cdot x = 0\}$ and $g_k(\bar{x}, \bar{y}) = g(x \cdot y, (-n_1)^{k-1})$ ($k > 0$). This is the nilpotent construction in [8, §3] up to shifts of the filtration.

4. MIXED FROBENIUS STRUCTURE

In this section, the base field is $K = \mathbb{C}$, a manifold means a complex manifold and vector bundles are assumed to be holomorphic. For a manifold M , T_M denotes the tangent bundle, \mathcal{T}_M its sheaf of local sections and we write $x \in \mathcal{T}_M$ to mean that x is a local section of T_M .

Although definitions here are for complex manifolds, they can be easily translated to C^∞ -manifolds ($K = \mathbb{R}$).

4.1. Saito structure. The following definition is due to Sabbah [11, Ch.VII].

Definition 4.1. Let M be a manifold. A Saito structure (without a metric) on M consists of

- a torsion-free flat connection ∇ on T_M ,
- an associative and commutative \mathcal{O}_M -bilinear multiplication \circ on \mathcal{T}_M with a global unit section e , and
- a global vector field E on M (called *the Euler vector field*),

satisfying the following conditions.

(i) The multiplication C_x by $x \in \mathcal{T}_M$ regarded as a local section of $\text{End } T_M$ satisfies

$$(4.1) \quad \nabla_x C_y - \nabla_y C_x = C_{[x,y]} ,$$

and the unit vector field e is flat, i.e. $\nabla e = 0$.

(ii) The vector field E satisfies $\nabla(\nabla E) = 0$ and

$$(4.2) \quad [E, x \circ y] - [E, x] \circ y - x \circ [E, y] = x \circ y \quad (x, y \in \mathcal{T}_M) .$$

In this article, we call a Saito structure without a metric a *Saito structure* for short.

Remark 4.2. In [11], a Saito structure is defined in terms of the symmetric Higgs field instead of the multiplication. As explained in [11, Ch.0.13], a symmetric Higgs field corresponds to a multiplication and Definition 4.1 is equivalent to that in *loc.cit.*.

Lemma 4.3. *Given a Saito structure (∇, \circ, E) on a manifold M , there exists a local vector field $\mathcal{G} \in \mathcal{T}_M$ such that*

$$(4.3) \quad \nabla_x \nabla_y \mathcal{G} = x \circ y$$

holds for any flat vector fields $x, y \in \mathcal{T}_M$. Moreover $\nabla \nabla([E, \mathcal{G}] - \mathcal{G}) = 0$.

We call \mathcal{G} satisfying eq.(4.3) a (local) potential vector field.

Proof. Let $\{t_\alpha\}_\alpha$ be a local coordinate system on M whose corresponding local frame fields $\{\partial_\alpha\}_\alpha$ are ∇ -flat. Let us write $\partial_\alpha \circ \partial_\beta = \sum_\gamma C_{\alpha\beta}^\gamma \partial_\gamma$. The commutativity implies $C_{\alpha\beta}^\gamma = C_{\beta\alpha}^\gamma$. Eq.(4.1) is equivalent to $\partial_\alpha C_{\beta\gamma}^\delta = \partial_\beta C_{\alpha\gamma}^\delta$. Therefore there exist $\mathcal{G}^\gamma \in \mathcal{O}_M$ such that $\partial_\alpha \partial_\beta \mathcal{G}^\gamma = C_{\alpha\beta}^\gamma$. Then $\mathcal{G} := \sum_\gamma \mathcal{G}^\gamma \partial_\gamma$ satisfies eq.(4.3). The second statement follows from eq.(4.2). \square

Remark 4.4. It is shown that Frobenius manifold structure defined by Dubrovin [2] is equivalent to a Saito structure with a metric [11, Ch.VII, Prop.2.2]. In the case when M is a Frobenius manifold, eq.(4.1) is equivalent to the potentiality condition, and the gradient vector field of the potential function is a potential vector field.

4.2. Mixed Frobenius structure.

Definition 4.5. A mixed Frobenius structure (MFS) on a manifold M consists of a Saito structure (∇, \circ, E) together with

- an increasing sequence of subbundles I_\bullet of T_M , and
- metrics (i.e. nondegenerate symmetric \mathcal{O}_M -bilinear forms) g_\bullet on $\mathcal{I}_\bullet/\mathcal{I}_{\bullet-1}$,

satisfying the following conditions.

- (i) $(\circ, I_\bullet, g_\bullet)$ is a mixed Frobenius algebra structure on T_M , i.e. \mathcal{I}_k are ideals of \mathcal{T}_M and all g_k 's are \circ -invariant.
- (ii) The subbundles I_k ($k \in \mathbb{Z}$) are preserved by ∇ and the metrics are compatible with ∇ , i.e.

$$(4.4) \quad zg_k(\overline{x}, \overline{y}) = g_k(\overline{\nabla_z x}, \overline{y}) + g_k(\overline{x}, \overline{\nabla_z y}) \quad (k \in \mathbb{Z}, z \in \mathcal{T}_M, x, y \in \mathcal{I}_k).$$

Here $x \mapsto \overline{x}$ denotes the projection $\mathcal{I}_k \rightarrow \mathcal{I}_k/\mathcal{I}_{k-1}$.

- (iii) The subbundles I_k ($k \in \mathbb{Z}$) are preserved by $[E, -]$ and there exists a collection of numbers $\{D_k \in K\}_{k \in \mathbb{Z}}$ (called *charges*) such that

$$(4.5) \quad Eg_k(\overline{x}, \overline{y}) - g_k(\overline{[E, x]}, \overline{y}) - g_k(\overline{x}, \overline{[E, y]}) = (2 - D_k)g_k(\overline{x}, \overline{y}) \quad (k \in \mathbb{Z}, x, y \in \mathcal{I}_k).$$

A MFS with the trivial filtration I_\bullet (i.e. $0 \subset T_M$) is the same as a Saito structure with a metric [11] and also the same as a Frobenius manifold structure [2].

Lemma 4.6. *If $(\nabla, \circ, E, I_\bullet, g_\bullet)$ is a MFS on a manifold M , then each $\mathcal{I}_k \subset \mathcal{T}_M$ ($k \in \mathbb{Z}$) is involutive.*

Proof. The lemma follows from the condition that I_k is preserved by the torsion free affine connection ∇ . \square

As a consequence, there exists a system of flat local coordinate system $\{t_{k\alpha}\}_{k \in \mathbb{Z}, 1 \leq \alpha \leq \dim I_k/I_{k-1}}$ such that $\{t_{k\alpha}\}_{k \leq l, 1 \leq \alpha \leq \dim I_k/I_{k-1}}$ is a local coordinate system of leaves of I_l .

Remark 4.7. The definition of MFS in this article is different from that in our previous article [8, Definition 6.2] in a few points.

Firstly charges $\{D_k\}$ are allowed to take any values. The merit is that any mixed Frobenius algebra has a MFS (see Proposition 4.8 below) whereas the condition $D_k = D_0 - k$ in the old definition is quite restrictive.

Secondly the compatibility conditions of the multiplication with the connection and the Euler vector field are strengthened as we adopt the Saito structure. (Compare [8, eqs.(6.2), (6.9)] with eqs.(4.1), (4.2)). The reason for this change is the existence of a local potential vector field (Lemma 4.3) and the flat meromorphic connection [11] on the

Saito structure. We believe that they may play important roles in formulating the local mirror symmetry as equivalence of MFS's (cf. §6.4).

4.3. An algebra with a Frobenius filtration has a MFS. Let $(A, I_\bullet, g_\bullet)$ be a mixed Frobenius algebra. We assume that $A = \bigoplus_{d \in \mathbb{Z}} A_d$ is a graded algebra satisfying $I_k = \bigoplus_d I_k \cap A_d$. Moreover we assume that there exist $\{D_k \in \mathbb{Z} \mid k \in \mathbb{Z}, I_k/I_{k-1} \neq 0\}$ such that

$$g_k(x, y) = 0 \quad \text{unless} \quad |x| + |y| = D_k.$$

Here $|x|$ denotes the degree of $x \in A$. Notice that any mixed Frobenius algebra satisfies this assumption with $A = A_0$ and $D_0 = 0$.

Let $\{e_{k\alpha} \mid k \in \mathbb{Z}, 1 \leq \alpha \leq \dim I_k/I_{k-1}\}$ be a homogeneous basis of A such that $\{e_{k\alpha} \mid k \leq l, 1 \leq \alpha \leq \dim I_l/I_{l-1}\}$ is a basis of I_l . Let $\{t_{k\alpha}\}$ be the associated coordinates of A .

Proposition 4.8. *The trivial connection d , the multiplication on A , the vector field*

$$E = \sum_{k, \alpha} (1 - |e_{k\alpha}|) t_{k\alpha} \partial_{k\alpha},$$

and the nondegenerate filtration (I_\bullet, g_\bullet) form a MFS on A of charges $\{D_k\}$.

5. FORMAL MIXED FROBENIUS STRUCTURE

In this section we will define a formal (and logarithmic) version of the MFS using [4] as reference.

In this section, the base field K may be any subfield of \mathbb{C} .

5.1. Notations. Fix a positive integer $n \in \mathbb{Z}_{>0}$ and a non-negative integer $m \in \mathbb{Z}_{\geq 0}$. We set $R = K[[t_1, \dots, t_n, q_1, \dots, q_m]]$ and let

$$(5.1) \quad P = \{\Phi(t, q) \in R \mid \exists d_1, \dots, d_m \in \mathbb{Z}_{\geq 0} \text{ s.t. } q_1^{-d_1} \cdots q_m^{-d_m} \Phi(t, q) \in R^\times\}$$

which is a submonoid of R . Let $M = \text{Spf } R$ be the formal completion of $K^{n+m} = \text{Spec } K[t, q]$ at the origin and let P_M be the constant sheaf on M with a stalk P . Denote by M^\dagger the formal scheme M equipped with the logarithmic structure associated to $P_M \hookrightarrow \mathcal{O}_M$.

Let \mathcal{T}_{M^\dagger} be the sheaf of logarithmic vector fields on M^\dagger which is freely generated by $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ and $q_1 \frac{\partial}{\partial q_1}, \dots, q_m \frac{\partial}{\partial q_m}$ over \mathcal{O}_M . Namely, if we let

$$(5.2) \quad H_K = \bigoplus_{\alpha=1}^n K \frac{\partial}{\partial t_\alpha} \oplus \bigoplus_{i=1}^m K q_i \frac{\partial}{\partial q_i},$$

then we have $\mathcal{T}_{M^\dagger} = \mathcal{O}_M \otimes_K H_K$. Define a flat connection ∇ on \mathcal{T}_{M^\dagger} by $\nabla = d \otimes 1_{H_K}$. The Lie bracket $[\ , \]$ satisfies

$$(5.3) \quad [x, y] = \nabla_x y - \nabla_y x \quad (x, y \in \mathcal{T}_{M^\dagger}) .$$

5.2. Formal mixed Frobenius structure. We keep the notations in §5.1.

Definition 5.1. A formal Saito structure on M^\dagger consists of

- an \mathcal{O}_M -bilinear multiplication \circ on \mathcal{T}_{M^\dagger} and
- an element $E \in \mathcal{T}_{M^\dagger}$

satisfying the following conditions.

(i) The multiplication \circ is compatible with ∇ in the sense that

$$(5.4) \quad \nabla_x(y \circ z) = \nabla_y(x \circ z) \quad (x, y, z \in H_K) ,$$

and the unit element e satisfies $\nabla e = 0$, i.e. $e \in H_K$.

(ii) The element E satisfies $\nabla_x \nabla_y E = 0^3$ for $x, y \in H_K$ and

$$(5.5) \quad [E, x \circ y] - x \circ [E, y] - [E, x] \circ y = x \circ y \quad (x, y \in \mathcal{T}_{M^\dagger}) .$$

If (\circ, E) is a formal Saito structure on M^\dagger , then as in Lemma 4.3, there exists $\mathcal{G} \in K[\log q_1, \dots, \log q_m] \otimes_K \mathcal{T}_{M^\dagger}$ satisfying eq.(4.3) for $x, y \in H_K$.

Definition 5.2. A formal mixed Frobenius structure on M^\dagger consists of

- a formal Saito structure (\circ, E) on M^\dagger , and
- a nondegenerate filtration (I_\bullet, g_\bullet) on H_K ,

satisfying the following conditions.

(i) $\mathcal{I}_k = \mathcal{O}_M \otimes_K I_k$ is an ideal and g_k extended \mathcal{O}_M -bilinearly to \mathcal{I}_k is \circ -invariant, i.e.

$$(5.6) \quad g_k(x \circ y, z) = g_k(y, x \circ z) \quad (x \in \mathcal{T}_{M^\dagger}, y, z \in \mathcal{I}_k) .$$

(ii) \mathcal{I}_k is preserved by $[E, -]$, i.e. $[E, x] \in \mathcal{I}_k$ ($x \in \mathcal{I}_k$) and there exists a collection of numbers $\{D_k \in K \mid k \in \mathbb{Z}, I_k/I_{k-1} \neq 0\}$, called *charges*, satisfying

$$(5.7) \quad E g_k(\bar{x}, \bar{y}) - g_k(\overline{[E, x]}, \bar{y}) - g_k(\bar{x}, \overline{[E, y]}) = (2 - D_k) g_k(\bar{x}, \bar{y}) \quad (x, y \in \mathcal{I}_k) .$$

Here $x \mapsto \bar{x}$ denotes the projection $\mathcal{I}_k \rightarrow \mathcal{I}_k/\mathcal{I}_{k-1}$.

Remark 5.3 (on the convergent case). Let (\circ, E) (resp. $(\circ, E, I_\bullet, g_\bullet)$) be a formal Saito structure (resp. a formal MFS) on M^\dagger and let $C_{\alpha\beta}^\gamma \in \mathcal{O}_M$ ($1 \leq \alpha, \beta, \gamma \leq n+m$) denote the structure constants of \circ with respect to the basis $(x_1, \dots, x_{n+m}) = (\partial_{t_1}, \dots, \partial_{t_n}, q_1 \partial_{q_1}, \dots, q_m \partial_{q_m})$. If there exists an open neighborhood U' of $0 \in K^{n+m} = \text{Spec } K[t, q]$ where all $C_{\alpha\beta}^\gamma$

³ If we write $E = \sum_{\alpha=1}^n E_\alpha \partial_{t_\alpha} + \sum_{i=1}^m E_i q_i \partial_{q_i}$, the condition $\nabla \nabla E = 0$ implies that E_α ($1 \leq \alpha \leq n$), and E_i ($1 \leq i \leq m$) are linear polynomials in t and independent of q .

converge, then (∇, \circ, E) (resp. $(\nabla, \circ, E, I_\bullet, g_\bullet)$) is a Saito structure (resp. a MFS) on $U = U' \cap \{q_1 \cdots q_m \neq 0\}$ with local flat coordinates $t_1, \dots, t_n, \log q_1, \dots, \log q_m$. In the case when the filtration I_\bullet is trivial, then $(U, \circ, E, e, g_\bullet)$ is a Frobenius manifold with logarithmic poles along the divisor $\{q_1 \cdots q_m = 0\}$ (see [10] for the definition).

5.3. Localized formal Frobenius structure over $K[\lambda]$. We still keep the notations in §5.1 and use superscripts λ for those tensored with $K[\lambda]$: $\mathcal{O}_M^\lambda := \mathcal{O}_M \otimes_K K[\lambda]$, $H_K^\lambda = H_K \otimes_K K[\lambda]$, and $\mathcal{T}_{M^\dagger}^\lambda := \mathcal{T}_{M^\dagger} \otimes_K K[\lambda] = \mathcal{O}_M \otimes_K H_K^\lambda$. We have a flat connection ∇ on $\mathcal{T}_{M^\dagger}^\lambda$ defined by the $K[\lambda]$ -linear extension of that introduced in §5.1.

Definition 5.4. A localized formal Frobenius structure over $K[\lambda]$ on M^\dagger consists of

- an \mathcal{O}_M^λ -bilinear multiplication $*$ on $\mathcal{T}_{M^\dagger}^\lambda$,
- an element $\mathbf{E} \in \mathcal{T}_{M^\dagger}^\lambda$, and
- a localized $K[\lambda]$ -metric g^λ on H_K^λ

satisfying the following conditions.

(i) The multiplication $*$ is compatible with ∇ in the sense that

$$(5.8) \quad \nabla_x(\mathbf{y} * \mathbf{z}) = \nabla_y(\mathbf{x} * \mathbf{z}) \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in H_K^\lambda),$$

and the unit \mathbf{e} satisfies $\nabla \mathbf{e} = 0$, i.e. $\mathbf{e} \in H_K^\lambda$.

(ii) The element \mathbf{E} satisfies $\nabla_x \nabla_y \mathbf{E} = 0$ for $\mathbf{x}, \mathbf{y} \in H_K^\lambda$ and

$$(5.9) \quad [\mathbf{E}^\lambda, \mathbf{x} * \mathbf{y}] - \mathbf{x} * [\mathbf{E}^\lambda, \mathbf{y}] - [\mathbf{E}^\lambda, \mathbf{x}] * \mathbf{y} = \mathbf{x} * \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathcal{T}_{M^\dagger}^\lambda),$$

where $\mathbf{E}^\lambda := \mathbf{E} + \lambda \frac{\partial}{\partial \lambda}$.

(iii) g^λ , extended \mathcal{O}_M^λ -bilinearly to $\mathcal{T}_{M^\dagger}^\lambda$, is $*$ -invariant.

(iv) There exists $D \in K$ (called a charge) satisfying

$$(5.10) \quad \mathbf{E}^\lambda g^\lambda(\mathbf{x}, \mathbf{y}) - g^\lambda([\mathbf{E}^\lambda, \mathbf{x}], \mathbf{y}) - g^\lambda(\mathbf{x}, [\mathbf{E}^\lambda, \mathbf{y}]) = (2 - D)g^\lambda(\mathbf{x}, \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathcal{T}_{M^\dagger}^\lambda).$$

Proposition 5.5. *Let $(*, \mathbf{E}, g^\lambda)$ be a localized formal Frobenius structure over $K[\lambda]$ of charge D on M^\dagger . Let \circ be the multiplication on \mathcal{T}_{M^\dagger} induced by $\pi : \mathcal{T}_{M^\dagger}^\lambda \rightarrow \mathcal{T}_{M^\dagger} = \mathcal{T}_{M^\dagger}^\lambda / \lambda \mathcal{T}_{M^\dagger}^\lambda$, $E = \pi(\mathbf{E})$, and let (I_\bullet, g_\bullet) be the nondegenerate filtration on H_K induced from the localized $K[\lambda]$ -metric g^λ (see Lemma 3.4). Then $(\circ, E, I_\bullet, g_\bullet)$ is a formal MFS on M^\dagger of charges $\{D_k = D - k\}$.*

Proof. First, the conditions (i) and (ii) of Definition 5.1 follow from the conditions (i) and (ii) in Definition 5.4 respectively.

The $*$ -invariance of g^λ implies that $\mathcal{O}_M \otimes I_k^\lambda =: \mathcal{I}_k^\lambda \subset \mathcal{T}_{M^\dagger}^\lambda$ is an ideal with respect to $*$, which in turn implies that \mathcal{I}_k is an ideal with respect to \circ . It also implies the \circ -invariance of the metrics g_\bullet .

Eq.(5.10) implies that the Lie bracket $[\mathbf{E}^\lambda, -]$ preserves \mathcal{I}_k^λ . From this it follows that $[E, -]$ preserves \mathcal{I}_k . Eq.(5.10) also implies the condition (5.7) as follows. For $x, y \in \mathcal{I}_k$, we have

$$\begin{aligned} Eg_k(\bar{x}, \bar{y}) &= \text{Res}_{\lambda=0} \lambda^{k-1} (k + \mathbf{E}^\lambda) g^\lambda(\mathbf{x}, \mathbf{y}) \\ &\stackrel{(5.10)}{=} kg_k(\bar{x}, \bar{y}) + \text{Res}_{\lambda=0} \lambda^{k-1} \{g^\lambda([\mathbf{E}^\lambda, \mathbf{x}], \mathbf{y}) + g^\lambda(\mathbf{x}, [\mathbf{E}^\lambda, \mathbf{y}]) + (2 - D)g^\lambda(\mathbf{x}, \mathbf{y})\} \\ &= (2 - D + k)g_k(\bar{x}, \bar{y}) + g_k([E, x], y) + g_k(x, [E, y]) , \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in \mathcal{I}_k^\lambda$ are lifts of x, y . \square

6. LOCAL QUANTUM COHOMOLOGY

In this section, K denotes either \mathbb{R} or \mathbb{C} .

6.1. Notations. Let X be a smooth complex projective variety. Let $\mathcal{V} \rightarrow X$ be a concave⁴ vector bundle of rank r . Let $S^1 = \text{U}(1)$ act on \mathcal{V} by the scalar multiplication on the fiber. The generator of the S^1 -equivariant cohomology of a point is denoted λ .

Let $H_K := H^{\text{even}}(X, K)$. We fix a basis $\{\phi_1, \dots, \phi_p\}$ of $H^2(X, \mathbb{Z})$ satisfying the condition that $\int_C \phi_i \geq 0$ for any curve $C \subset X$.⁵ We also fix a homogeneous basis $\{\phi_0 = 1, \phi_1, \dots, \phi_p, \phi_{p+1}, \dots, \phi_s\}$ of H_K .

Let t_0, \dots, t_s be the coordinates on H_K associated to the basis. We set $R = K[[t, q]]$ where $t = (t_0, t_{p+1}, \dots, t_s)$ and $q = (q_1, \dots, q_p)$ with $q_i = e^{t_i}$. As in §5.1, we consider the formal scheme $M = \text{Spf } R$ with a fixed logarithmic structure defined by the monoid (5.1) and denote it by M^\dagger . We identify H_K with the linear space of derivations on R defined in eq.(5.2) by

$$(6.1) \quad \begin{cases} \phi_\alpha \mapsto \frac{\partial}{\partial t_\alpha} & (\alpha = 0, p+1, \dots, s) \\ \phi_i \mapsto q_i \frac{\partial}{\partial q_i} & (1 \leq i \leq p) \end{cases} .$$

Hence $\mathcal{T}_{M^\dagger} = \mathcal{O}_M \otimes_K H_K$. The same notations \mathcal{O}_M^λ , H_K^λ and $\mathcal{T}_{M^\dagger}^\lambda$ as in §5.3 will be used.

We put the grading on the vector space H_K by setting $|\phi| = k$ if $\phi \in H^{2k}(X, K)$. We also put the gradings on the rings \mathcal{O}_M and \mathcal{O}_M^λ by $|t_\alpha| = 1 - |\phi_\alpha|$ ($\alpha = 0, p+1, \dots, s$), $|\lambda| = 1$ and $|q_i| = \xi_i$, where ξ_i are defined by

$$(6.2) \quad c_1(X) + c_1(\mathcal{V}) = \sum_{i=1}^p \xi_i \phi_i .$$

⁴A vector bundle \mathcal{V} is concave if $H^0(C, f^*\mathcal{V}) = 0$ for any genus zero stable map (f, C) to X of non-zero degree.

⁵The existence of such a basis follows from the fact that the Mori cone $\overline{NE}_{\mathbb{R}}(X)$ of a smooth projective variety X does not contain a straight line (see e.g. [6, Corollary 1.19]). If σ denotes the image of $\overline{NE}_{\mathbb{R}}(X)$ in $H_2(X, \mathbb{R})$, the dual cone $\sigma^\vee = \{x \in H^2(X, \mathbb{R}) \mid \langle x, y \rangle \geq 0, y \in \sigma\}$ is of maximal dimension. Therefore there exists an integral basis ϕ_1, \dots, ϕ_p of $H^2(X, \mathbb{R})$ such that $\phi_i \in \sigma^\vee$.

Then we have the induced gradings on \mathcal{T}_{M^\dagger} and $\mathcal{T}_{M^\dagger}^\lambda$.

Let

$$(6.3) \quad \mathbf{E} = \sum_{\alpha=0}^s (1 - |\phi_\alpha|) t_\alpha \frac{\partial}{\partial t_\alpha} + \sum_{i=1}^p \xi_i q_i \frac{\partial}{\partial q_i}, \quad \mathbf{E}^\lambda = \mathbf{E} + \lambda \frac{\partial}{\partial \lambda}.$$

Then, for a homogeneous $f \in \mathcal{O}_M^\lambda$ and $\mathbf{x} \in \mathcal{T}_{M^\dagger}^\lambda$, we have

$$(6.4) \quad \mathbf{E}^\lambda f = |f|f, \quad [\mathbf{E}^\lambda, \mathbf{x}] = (|\mathbf{x}| - 1)\mathbf{x}.$$

6.2. Localized formal Frobenius structure over $K[\lambda]$. The following material can be found in [3]. Let g^λ be a localized $K[\lambda]$ -metric on H_K^λ defined by

$$(6.5) \quad g^\lambda(\phi, \varphi) = \int_X \phi \cup \varphi \cup \frac{1}{e_{S^1}(\mathcal{V})}$$

where $e_{S^1}(\mathcal{V})$ is the S^1 -equivariant Euler class of \mathcal{V} :

$$e_{S^1}(\mathcal{V}) = \lambda^r + c_1(\mathcal{V})\lambda^{r-1} + \cdots + c_r(\mathcal{V}).$$

Lemma 6.1. g^λ and \mathbf{E} (in eq.(6.3)) satisfy eq.(5.10) with $D = \dim_{\mathbb{C}} X + r$.

Proof. By the degree consideration, g^λ satisfies

$$(6.6) \quad g^\lambda(\phi_\alpha, \phi_\beta) = \eta_{\alpha\beta} \lambda^{|\phi_\alpha| + |\phi_\beta| - \dim_{\mathbb{C}} X - r} \quad (\eta_{\alpha\beta} \in K).$$

This together with eq.(6.4) implies the lemma. \square

We define the multiplication on $\mathcal{T}_{M^\dagger}^\lambda$ as follows. For $x_1, \dots, x_m \in H_K$ and $d \in H_2(X, \mathbb{Z})$, let

$$(6.7) \quad \langle x_1, \dots, x_m \rangle_{\mathcal{V}, d} = \int_{[\overline{M}_{0,m}(X, d)]^{\text{vir}}} \prod_{i=1}^m ev_i^* x_i \cup e_{S^1}(-R^\bullet \mu_* ev_{m+1}^* \mathcal{V}) \in K[\lambda]$$

where $\overline{M}_{0,m}(X, d)$ is the moduli stack of genus zero stable maps to X of degree d with m marked points, $ev_i : \overline{M}_{0,m}(X, d) \rightarrow X$ is the evaluation map at the i th marked point, and $\mu : \overline{M}_{0,m+1}(X, d) \rightarrow \overline{M}_{0,m}(X, d)$ is the forgetful map. We define the multiplication $*_{\mathcal{V}}$ on $\mathcal{T}_{M^\dagger}^\lambda$ by

$$(6.8) \quad \begin{aligned} g^\lambda(\mathbf{x} *_{\mathcal{V}} \mathbf{y}, \mathbf{z}) &= \sum_d \sum_{m \geq 0} \frac{1}{m!} \langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \underbrace{\tau, \dots, \tau}_m \rangle_{\mathcal{V}, d} \quad (\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{T}_{M^\dagger}^\lambda) \\ &= \sum_d \sum_{m \geq 0} \frac{1}{m!} \langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \underbrace{\tau_{\geq 4}, \dots, \tau_{\geq 4}}_m \rangle_{\mathcal{V}, d} q^d. \end{aligned}$$

In the first line, $\tau = \sum_{\alpha=0}^s t_\alpha \phi_\alpha$ and in the second line, $\tau_{\geq 4} = \sum_{\alpha=p+1}^s t_\alpha \phi_\alpha$ and $q^d = e_{\int d t_1 \phi_1 + \cdots + t_p \phi_p}$. In passing to the second line, the fundamental class axiom and the divisor axiom of Gromov–Witten theory (see, e.g., [9, III, §5]) are used.

Lemma 6.2. $(\mathcal{T}_{M^\dagger}^\lambda, *_\mathcal{V})$ is a graded ring. Hence the multiplication $*_\mathcal{V}$ and \mathbf{E} in eq.(6.3) satisfy eq.(5.9).

Proof. The lemma follows from the degree axiom of Gromov–Witten theory. \square

Proposition 6.3. $(g^\lambda, *_\mathcal{V}, \mathbf{E})$ is a localized formal Frobenius structure over $K[\lambda]$ of charge $\dim_{\mathbb{C}} X + r$ on M^\dagger .

Proof. By the definition of $*_\mathcal{V}$, it is clear that g^λ is $*_\mathcal{V}$ -invariant and satisfies eq.(5.8). \square

6.3. Formal mixed Frobenius structure from local quantum cohomology.

Theorem 6.4. The collection $(\circ_\mathcal{V}, E, I_\bullet, g_\bullet)$ of the following data determines a formal MFS of charges $\{\dim_{\mathbb{C}} X + r - k\}_{k \in \mathbb{Z}}$ on M^\dagger ;

- the multiplication $\circ_\mathcal{V}$ on \mathcal{T}_{M^\dagger} induced from the multiplication $*_\mathcal{V}$ on $\mathcal{T}_{M^\dagger}^\lambda$,
- the Euler vector field E which has the same expression as \mathbf{E} in eq.(6.3),
- a nondegenerate filtration (I_\bullet, g_\bullet) on H_K constructed by Lemma 3.4.

Proof. Applying Proposition 5.5 to the localized formal Frobenius structure over $K[\lambda]$ in Proposition 6.3, we obtain the result. \square

Remark 6.5 (on convergence of the formal MFS). If $\mathcal{V} \rightarrow X$ is a negative line bundle, it can be shown that the structure constants of $\circ_\mathcal{V}$ are convergent if those of the quantum product of X are convergent e.g. if X is a smooth projective toric variety [5]. The proof is completely the same as Iritani’s [5] except that it is necessary to modify the proof of his Lemma 4.2. For a pair of such X and a negative line bundle \mathcal{V} , the formal MFS described in this subsection is actually a MFS on some open subset of H_K . (See Remark 5.3).

Let us describe the MFS in Theorem 6.4 concretely. The multiplication $\circ_\mathcal{V}$ on \mathcal{T}_{M^\dagger} is written as follows. For $d \neq 0$, $x_1, \dots, x_m \in H_K$, let

$$(6.9) \quad \langle x_1, \dots, x_m \rangle_{\mathcal{V}, d}^{\lambda=0} = \int_{[\overline{M}_{0,m}(X, d)]^{\text{vir}}} \prod_{i=1}^m ev_i^* x_i \cup e(R^1 \mu_* ev_{m+1}^* \mathcal{V}),$$

where e denotes the (non-equivariant) Euler class. Then a potential vector field \mathcal{G} for $\circ_\mathcal{V}$ (cf. Lemma 4.3) is given by

$$(6.10) \quad \mathcal{G} = \sum_{\alpha=0}^s \left(\partial_\alpha \Phi_{\text{cl}} \right) \phi^\alpha + \sum_{\alpha=1}^s \left(\partial_\alpha \Phi_{\text{qu}} \right) c_r(\mathcal{V}) \cup \phi^\alpha,$$

where $\partial_\alpha = \frac{\partial}{\partial t_\alpha}$,

$$\Phi_{\text{cl}} = \frac{1}{3!} \int_X \tau \cup \tau \cup \tau, \quad \Phi_{\text{qu}} = \sum_{d \neq 0} \sum_{m \geq 0} \frac{q^d}{m!} \underbrace{\langle \tau_{\geq 4}, \dots, \tau_{\geq 4} \rangle_{\mathcal{V}, d}^{\lambda=0}}_m,$$

and $\{\phi^\alpha\}$ is a basis of H_K dual to $\{\phi_\alpha\}$ with respect to the intersection form of X .

By the result of §3.2, the nondegenerate filtration (I_\bullet, g_\bullet) on H_K is given as follows.

$$(6.11) \quad \begin{aligned} I_k &= 0 \quad (k < 0), \\ I_0 &= \{x \cup c_r(\mathcal{V}) \mid x \in H_K\}, \\ I_k &= I_0 + J_k, \quad J_k = p_r(\text{Ker } N^k), \end{aligned}$$

where

$$N = \begin{pmatrix} -c_1(\mathcal{V}) & 1 & 0 & \cdots & 0 \\ -c_2(\mathcal{V}) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{r-1}(\mathcal{V}) & 0 & 0 & \cdots & 1 \\ -c_r(\mathcal{V}) & 0 & 0 & \cdots & 0 \end{pmatrix} : H_K^{\oplus r} \rightarrow H_K^{\oplus r}$$

and p_r is the projection to the r th factor. The metrics g_k on I_k/I_{k-1} are given by

$$(6.12) \quad \begin{aligned} g_0(c_r(\mathcal{V}) \cup x, c_r(\mathcal{V}) \cup y) &= \int_X c_r(\mathcal{V}) \cup x \cup y \quad (x, y \in H_K), \\ g_k(\bar{x}, \bar{y}) &= \int_X x \cup p_1(N^{k-1}\bar{y}) \quad (k > 0, x, y \in J_k), \end{aligned}$$

where $\bar{y} \in \text{Ker } N^k$ is any lift of y .

Remark 6.6 (on the nilradical of $\circ_{\mathcal{V}}$). If (X, \mathcal{V}) satisfies the condition that $\int_C (c_1(X) + c_1(\mathcal{V})) \leq 0$ holds for any curve $C \subset X$, then $\phi_\alpha \circ_{\mathcal{V}} \phi_\beta \in \mathcal{O}_M \otimes_K H^{\geq |\phi_\alpha| + |\phi_\beta|}(X, K)$ holds by the degree axiom. Therefore for such (X, \mathcal{V}) , the nilradical of $(\mathcal{T}_{M^\dagger}, \circ_{\mathcal{V}})$ is $\mathcal{O}_M \otimes_K H^{\geq 2}(X, K)$.

6.4. Remarks on local mirror symmetry. Let X be a Fano toric surface and $\mathcal{V} = K_X$ be the canonical bundle. Take $\phi_{p+1} = \phi^0$. Then

$$\mathcal{G} = \sum_{\alpha=0}^{p+1} (\partial_\alpha \Phi_{\text{cl}}) \phi^\alpha + \sum_{i=1}^p k_i (\partial_i \Phi_{\text{qu}}) \phi_{p+1}$$

where

$$\Phi_{\text{qu}} = \sum_{d \neq 0} N_d q^d, \quad N_d = \int_{[\bar{M}_{0,0}(X,d)]^{\text{vir}}} e(R^1 \mu_* ev_{m+1}^* K_X),$$

and k_i are defined by $\sum_{i=1}^p k_i \phi_i = c_1(K_X)$. The coefficient of ϕ_{p+1} in the above \mathcal{G} is nothing but the function $\mathcal{F}_{\text{local}}$ in [1, §6.3].

Next, let us discuss the relationship with the mirror side of the story. Let Δ be the fan polytope of X . There is a certain family of curves $\mathcal{C} \rightarrow \mathcal{M}(\Delta)$ in $(\mathbb{C}^*)^2$ associated to Δ . It was shown that

$$H^*(X, \mathbb{C}) \cong H^2((\mathbb{C}^*)^2, C_z) \quad (z \in \mathcal{M}(\Delta))$$

as \mathbb{C} -vector spaces and that the weight filtration of the mixed Hodge structure on $H^2((\mathbb{C}^*)^2, C_z)$ coincides with Frobenius filtration (up to shifts). Compare [7, §8] with eq.(6.11) and [8, eq.(8.8)].

Under the mirror map, \mathcal{F}_{local} corresponds to a double logarithmic period of $\omega_0(z) = [(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0)] \in H^2((\mathbb{C}^*)^2, C_z)$ and $\{g_0(\phi_i \circ_{K_X} \phi_j, c_1(K_X))\}_{1 \leq i, j \leq p}$ is essentially equal to the Yukawa coupling defined in [7, §6].

It is desirable to construct a MFS on $H^2((\mathbb{C}^*)^2, C_z)$ which is compatible with its variation of mixed Hodge structures and which agrees with the MFS on $H^*(X, \mathbb{C})$ under the mirror map.

REFERENCES

- [1] Chiang, T.-M.; Klemm, A.; Yau, S.-T.; Zaslow, E., *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), no. 3, 495–565.
- [2] Dubrovin, Boris, *Geometry of 2D topological field theories*, in Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math. 1620, Springer, Berlin, 1996.
- [3] Givental, Alexander, *Elliptic Gromov–Witten invariants and the generalized mirror conjecture*, in Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107–155, World Sci. Publ., River Edge, NJ, 1998.
- [4] Gross, Mark, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, 114. American Mathematical Society, Providence, RI, 2011. xvi+317 pp.
- [5] Iritani, Hiroshi, *Convergence of quantum cohomology by quantum Lefschetz*, J. Reine Angew. Math. **610** (2007), 29–69.
- [6] Kollár, János; Mori, Shigefumi, *Birational geometry of algebraic varieties*. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998. viii+254 pp.
- [7] Konishi, Yukiko; Minabe, Satoshi, *Local B-model and mixed Hodge structure*, Adv. Theor. Math. Phys. **14** (2010), no. 4, 1089–1145.
- [8] ———, *Mixed Frobenius Structure and local A-model*, Preprint 2012, arXiv:1209.5550.
- [9] Manin, Yuri I. *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, 47. American Mathematical Society, Providence, RI, 1999. xiv+303 pp.
- [10] Reichelt, Thomas, *A construction of Frobenius manifolds with logarithmic poles and applications*, Comm. Math. Phys. **287** (2009), no. 3, 1145–1187.
- [11] Sabbah, Claude, *Déformations isomonodromiques et variétés de Frobenius*, EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2002. xvi+289 pp.

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